

Polar Codes and Polar Lattices for the Heegard-Berger Problem

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Abstract—Explicit coding schemes are proposed to achieve the rate-distortion bound for the Heegard-Berger problem using polar codes. Specifically, a nested polar code construction is employed to achieve the rate-distortion bound for the binary case. The nested structure contains two optimal polar codes for lossy source coding and channel coding, respectively. Moreover, a similar nested polar lattice construction is employed for the Gaussian case. The proposed polar lattice is constructed by nesting a quantization polar lattice and an AWGN capacity-achieving polar lattice.

I. INTRODUCTION

The well-known Wyner-Ziv problem is a lossy source coding problem in which a source sequence is compressed to be reconstructed in the presence of a correlated side information at the decoder [1]. An interesting question is whether a reconstruction with a non-trivial distortion quality can still be obtained in the absence of the side information at the receiver. The equivalent coding system for this problem contains two decoders, one with the side information, and the other without, as shown by Fig. 1.

In 1985, Heegard and Berger [2] characterized the rate-distortion function $R_{HB}(D_1, D_2)$ for this scenario, where D_1 is the distortion achieved without side information, D_2 is the distortion achieved with it, and $R_{HB}(D_1, D_2)$ denotes the minimum rate required to achieve the distortion pair (D_1, D_2) . They also gave an explicit expression for the quadratic Gaussian case. Kerpez [3] provided upper and lower bounds on the Heegard-Berger rate-distortion (HBRD) function for the binary case. The explicit expression for $R_{HB}(D_1, D_2)$ in the binary case was derived in [4] together with the corresponding optimal test channel. Our goal in this paper is to propose explicit coding schemes that can achieve the HBRD function for binary and Gaussian distributions.

Polar codes, proposed by Arıkan in [5], have been widely studied due to their achievability of the Shannon bounds in both channel and source coding problems, with low complexity. Polar codes are optimal for lossy source coding, as well as the Wyner-Ziv problem, for binary sources [6]. The optimality of polar codes for lossy compression of nonuniform sources is shown in [7]. For Gaussian sources, a polar lattice to achieve both the classical and Wyner-Ziv rate-distortion functions is proposed in [8]. Practical codes for the Gaussian Heegard-Berger problem were developed in [9] which hybridize trellis codes and low-density parity-check codes. Here, we show

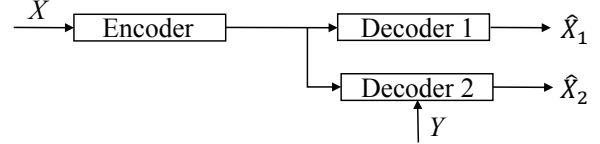


Fig. 1. Illustration of the Heegard-Berger rate-distortion problem.

that polar codes and polar lattices can achieve the theoretical performance bound in the Heegard-Berger problem.

II. PROBLEM STATEMENT

A. Heegard-Berger Problem

Let $(\mathcal{X}, \mathcal{Y}, P_{XY})$ be discrete memoryless sources denoted by random variables X and Y with a generic joint distribution P_{XY} over finite alphabets \mathcal{X} and \mathcal{Y} .

Definition 1. An (n, M, D_1, D_2) Heegard-Berger code for source X with side information Y consists of an encoder $f^{(n)} : \mathcal{X}^n \rightarrow I_M$ and two decoders $g_1^{(n)} : I_M \rightarrow \hat{\mathcal{X}}_1^n$; $g_2^{(n)} : I_M \times \mathcal{Y}^n \rightarrow \hat{\mathcal{X}}_2^n$, where $I_M \triangleq \{1, 2, \dots, M\}$ and $\hat{\mathcal{X}}_1, \hat{\mathcal{X}}_2$ are finite reconstruction alphabets, such that

$$\mathbb{E} \left[\frac{1}{n} \sum_{j=1}^n d(X_j, \hat{X}_{ij}) \right] \leq D_i, \quad i = 1, 2,$$

where \mathbb{E} is the expectation operation, and $d(\cdot, \cdot) < \infty$ is a per-letter distortion measure. In this paper, for binary sources, we set $d(\cdot, \cdot)$ as the Hamming distortion; and for Gaussian sources as the squared error distortion.

Definition 2. Rate R is said to be $\{(D_1, D_2) - \text{achievable}\}$, if for every $\epsilon > 0$ and sufficiently large n there exists an $(n, M, D_1 + \epsilon, D_2 + \epsilon)$ code with $R + \epsilon \geq \frac{1}{n} \log M$.

The HBRD function, $R_{HB}(D_1, D_2)$, is defined as the infimum of (D_1, D_2) -achievable rates. A single-letter expression for $R_{HB}(D_1, D_2)$ is given in the following theorem.

Theorem 3. ([2, Theorem 1])

$$R_{HB}(D_1, D_2) = \min_{(U_1, U_2) \in \mathcal{P}(D_1, D_2)} [I(X; U_1) + I(X; U_2 | U_1, Y)],$$

where $\mathcal{P}(D_1, D_2)$ is the set of all auxiliary random variables $(U_1, U_2) \in \mathcal{U}_1 \times \mathcal{U}_2$ jointly distributed with the generic random variables (X, Y) , such that: i) $Y \leftrightarrow X \leftrightarrow (U_1, U_2)$ forms a Markov chain; ii) $|\mathcal{U}_1| \leq |\mathcal{X}| + 2$ and $|\mathcal{U}_2| \leq (|\mathcal{X}| + 1)^2$; iii)



Fig. 2. The optimal forward test channel for the region described in Corollary 4. The crossover probability η for the BSC between U_2 and U_1 satisfies $D_2 * \eta = D_1$.

there exist functions φ_1 and φ_2 such that $\mathbb{E}d(X, \varphi_1(U_1)) \leq D_1$ and $\mathbb{E}d(X, \varphi_2(U_1, U_2, Y)) \leq D_2$.

B. Doubly Symmetric Binary Sources (DSBS)

Let X be a binary discrete memoryless source (DMS) with uniform distribution, i.e., $\mathcal{X} = \{0, 1\}$. The binary side information is specified by $Y = X \oplus Z$, where Z is a Bernoulli random variable with $P_Z(z = 1) = p < 0.5$, and \oplus denotes modulo two addition.

It has been shown in [3] that the HBRD function for DSBS can be characterized over four regions. Region I is defined by $0 \leq D_1 < 0.5$ and $0 \leq D_2 < \min(D_1, p)$. This is a non-degenerate region so that $R_{HB}(D_1, D_2)$ is a function of D_1 and D_2 ; Region II is defined by $D_1 \geq 0.5$ and $0 \leq D_2 \leq p$. In this region the problem degenerates to Wyner-Ziv encoding for the second decoder; Region III is defined by $0 \leq D_1 \leq 0.5$ and $D_2 \geq \min(D_1, p)$, in which the problem degenerates to ordinary lossy compression for the first decoder; Region IV is defined by $D_1 > 0.5$ and $D_2 > p$, in which the distortion constraints are trivially met. The HBRD function in Regions II and III can be trivially achieved by using polar codes as described in [6].

Here we focus on the non-degenerate Region I. The explicit expression of the rate-distortion function and the corresponding optimal test channel for Region I are given by [4].

First, we need the following function from [1] that is defined on the domain $0 \leq u \leq 1$, $G(u) \triangleq h(p * u) - h(u)$, where $h(u)$ is the binary entropy function $h(u) \triangleq -u \log u - (1 - u) \log(1 - u)$, and $p * u$ is the binary convolution for $0 \leq p, u \leq 1$, $p * u \triangleq p(1 - u) + u(1 - p)$. Next, recall the definition of the critical distortion d_c in the Wyner-Ziv problem for DSBS [1], for which $\frac{G(d_c)}{d_c - p} = G'(d_c)$.

The following corollary from [4] specifies a simple forward test channel for the region $D_2 \leq \min(d_c, D_1)$ and $D_1 \leq 0.5$. This region belongs to Region I, and contains smaller distortion pairs than the rest of Region I.

Corollary 4. ([4, Corollary 2]) For distortion pairs (D_1, D_2) satisfying $D_1 \leq 0.5$ and $D_2 \leq \min(d_c, D_1)$, we have

$$R_{HB}(D_1, D_2) = 1 - h(D_1 * p) + G(D_2). \quad (1)$$

From [4], the optimal forward test channel for this case is given as a cascade of two binary symmetric channels (BSCs), as depicted in Fig. 2.

In Section III, we propose a polar code design that achieves the HBRD function in the region specified by Corollary 4, as well as another polar code scheme for the rest of Region I.

C. Gaussian Sources

Suppose $Y = X + Z$, where X and Z are independent (zero-mean) Gaussian random variables with variances σ_X^2 and σ_Z^2 ,

respectively, i.e., $X \sim \mathcal{N}(0, \sigma_X^2)$ and $Z \sim \mathcal{N}(0, \sigma_Z^2)$. The explicit expression for $R_{HB}(D_1, D_2)$ in this case is given in [2]. The optimal test channels are given by $X = U_1 + Z_1$ and $X = U_2 + Z_2$, where Z , Z_1 and Z_2 are independent zero-mean Gaussian random variables. We have $Z_i \sim \mathcal{N}(0, D_i)$, $i = 1, 2$.

Decoder 1 receives only U_1 , and reconstructs $\hat{X}_1 = \delta U_1$. Decoder 2 forms the optimum estimate of X given (U_1, U_2, Y) , which is a linear combination of the form $\hat{X}_2 = \alpha U_1 + \beta U_2 + \omega Y$. The minimum of $I(X; U_1) + I(X; U_2 | U_1, Y)$ subject to $\mathbb{E}(X - \hat{X}_i)^2 = D_i$, for $i = 1, 2$, is achieved when $I(X; U_1) = I(X; \hat{X}_1) = \frac{1}{2} \log(\sigma_X^2 / D_1)$, and $I(X; U_2 | U_1, Y) = I(X; \hat{X}_2 | U_1, Y) = \frac{1}{2} \log(\sigma_{X|Y, U_1}^2 / D_2)$, where $\sigma_{X|Y, U_1}^2 = \frac{D_1 \sigma_Z^2}{D_1 + \sigma_Z^2}$. It follows that [2]

$$R_{HB}(D_1, D_2) = \begin{cases} \frac{1}{2} \log \left(\frac{\sigma_X^2 \sigma_Z^2}{D_2 (D_1 + \sigma_Z^2)} \right), & \text{if } D_1 \leq \sigma_X^2, D_2 \leq \frac{D_1 \sigma_Z^2}{D_1 + \sigma_Z^2} \\ \frac{1}{2} \log \left(\frac{\sigma_X^2}{D_1} \right), & \text{if } D_1 \leq \sigma_X^2, D_2 \geq \frac{D_1 \sigma_Z^2}{D_1 + \sigma_Z^2} \\ \frac{1}{2} \log \left(\frac{\sigma_X^2 \sigma_Z^2}{D_2 (\sigma_X^2 + \sigma_Z^2)} \right), & \text{if } D_1 > \sigma_X^2, D_2 \leq \frac{D_1 \sigma_Z^2}{D_1 + \sigma_Z^2} \\ 0, & \text{if } D_1 > \sigma_X^2, D_2 \geq \frac{D_1 \sigma_Z^2}{D_1 + \sigma_Z^2}. \end{cases} \quad (2)$$

Similarly to the binary case, the first region in (2) is the only non-degenerate one. In the second region, the problem degenerates to ordinary lossy compression for Decoder 1. In the third region, the problem is equivalent to Wyner-Ziv coding for Decoder 2. The distortion levels in the last region can be trivially met. Polar lattice code constructions that meet the classical and Wyner-Ziv rate-distortion functions for Gaussian sources are given in [8].

III. POLAR CODES FOR DSBS

In this section, we utilize polar codes to achieve $R_{HB}(D_1, D_2)$ for DSBS in the region specified by Corollary 4. First, we give a brief overview of polar codes.

We let $G_2 = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$ be the generator matrix of polar codes with length $N = 2^n$ defined by $G_N = G_2^{\otimes n}$, where ' \otimes ' denotes the Kronecker product. A polar code $\mathcal{C}_N(F, u_F)$ is a linear code defined by [6] $\mathcal{C}_N(F, u_F) = \left\{ v_1^N G_N : v_F = u_F, v_{F^c} = \{0, 1\}^{|F^c|} \right\}$, for any $F \subseteq \{1, \dots, N\}$ and $u_F \in \{0, 1\}^{|F|}$. F is referred to as the frozen set. v_1^N denotes the vector (v_1, \dots, v_N) . u_F denotes the subvector $\{u_i\}_{i \in F}$. F^c and $|F|$ denote the complement and cardinality of F , respectively. The code $\mathcal{C}_N(F, u_F)$ is constructed by fixing u_F and varying the values in F^c .

A. Proposed Coding Scheme

In the above problem, Decoder 1 needs to achieve D_1 without side information. Therefore the minimum rate for Decoder 1 will be $R_1 = I(U_1; X) = 1 - h(D_1)$. It is known that polar codes are optimal for lossy source coding for

the binary symmetric source (BSS), as stated in the following theorem.

Theorem 5. ([10, Theorem 3]) *Let X be a BSS. For any rate $R > 1 - h(D_1)$ and any $0 < \beta < \frac{1}{2}$, there exists a sequence of polar codes of length N with rates $R_N < R$, which, under successive cancellation (SC) encoding and randomized rounding, achieve an expected distortion not greater than $D_1 + \mathcal{O}(2^{-N^\beta})$.*

An explicit polar code construction for this scheme is provided in [10]. From Theorem 5, we know that Decoder 1 can recover a reconstruction that is asymptotically close to U_1 when N is sufficiently large. Therefore, for simplicity, we assume that both Decoder 1 and Decoder 2 can recover U_1 in the following.

Decoder 2 observes the side information Y , in addition to U_1 that is reconstructed from the message for Decoder 1. Hence, both Y and U_1 can be considered as side information for Decoder 2 to achieve distortion D_2 . Therefore, the problem at Decoder 2 is very similar to Wyner-Ziv coding except that the decoder observes extra side information.

Achieving the Wyner-Ziv rate-distortion function using polar codes is based on the nested code structure proposed in [6]. Consider the Wyner-Ziv problem of compressing source X in the presence of side information Y using polar codes. The code \mathcal{C}_s (the corresponding frozen set is F_s) is designed to be a good source code for distortion D_2 . Further, code \mathcal{C}_c (the corresponding frozen set is F_c) is designed to be a good channel code for BSC($D_2 * p$). It has been shown in [6] that $F_c \supseteq F_s$, because the test channel BSC($D_2 * p$) is degraded with respect to BSC(D_2). In this case, the encoder transmits the sub-vector that belongs to the index set $F_c \setminus F_s$ to the decoder. The optimality of this scheme is proved in [6].

Similarly, the optimal rate-distortion performance for Decoder 2 in the Heegard-Berger problem can also be achieved by using nested polar codes. For $(U_1, U_2) \in \mathcal{P}(D_1, D_2)$,

$$\begin{aligned} I(X; U_2 | U_1, Y) &= I(U_2; X, U_1, Y) - I(U_2; Y, U_1) \\ &= I(U_2; X, U_1) - I(U_2; Y, U_1). \end{aligned} \quad (3)$$

The second equality holds since $Y \leftrightarrow X \leftrightarrow (U_1, U_2)$ forms a Markov chain. According to (3), the code \mathcal{C}_{s_2} (the corresponding frozen set is F_{s_2}) is designed to be a good source code for sources (X, U_1) with reconstruction U_2 . T_s denotes the test channel for this source code. Additionally, code \mathcal{C}_{c_2} (the corresponding frozen set is F_{c_2}) is designed to be a good channel code for the test channel T_c with input U_2 and output (Y, U_1) . To show the nested structure between \mathcal{C}_{s_2} and \mathcal{C}_{c_2} , we need to show that T_c is stochastically degraded with respect to T_s , according to [6, Lemma 4.7].

Proposition 6. $T_c : U_2 \rightarrow (Y, U_1)$ is stochastically degraded with respect to $T_s : U_2 \rightarrow (X, U_1)$, for random variables (X, Y, U_1, U_2) that agree with the forward test channel as shown in Fig. 2.

Proof: From the test channel structure, $Y \leftrightarrow X \leftrightarrow U_2 \leftrightarrow U_1$ forms a Markov chain. By definition,

$P_{X, U_1 | U_2}(x, u_1 | u_2) = P_{X | U_2}(x | u_2) P_{U_1 | U_2}(u_1 | u_2)$. We also have

$$\begin{aligned} &P_{Y, U_1 | U_2}(y, u_1 | u_2) \\ &= P_{Y | U_2}(y | u_2) P_{U_1 | U_2}(u_1 | u_2) \\ &= \sum_x P_{X, Y | U_2}(x, y | u_2) P_{U_1 | U_2}(u_1 | u_2) \\ &= \sum_x P_{X | U_2}(x | u_2) P_{Y | X, U_2}(y | x, u_2) P_{U_1 | U_2}(u_1 | u_2) \\ &= \sum_x P_{X | U_2}(x | u_2) P_{U_1 | U_2}(u_1 | u_2) P_{Y | X}(y | x) \\ &= \sum_x P_{X, U_1 | U_2}(x, u_1 | u_2) P_{Y | X}(y | x), \end{aligned}$$

completing the proof. \blacksquare

Therefore, we can claim that $F_{c_2} \supseteq F_{s_2}$. Rather than sending the entire vector that belongs to the index set $F_{s_2}^c$, the encoder sends only the sub-vector that belongs to $F_{c_2} \setminus F_{s_2}$ to Decoder 2, since Decoder 2 can extract some information on U_2 from the side information (U_1, Y) .

In summary, the coding scheme for Heegard-Berger problem for the case of Corollary 4 is as follows:

Encoding: The encoder applies lossy compression to source X with reconstruction U_1 and distortion D_1 . We construct the code $\mathcal{C}_{s_1} = \mathcal{C}_N(F_{s_1}, \bar{0}) = \{w_1^N G_N : w_{F_{s_1}} = \bar{0}, w_{F_{s_1}^c} = \{0, 1\}^{|F_{s_1}^c|}\}$, and the encoder transmits the compressed sequence $w_{F_{s_1}^c}$ to the decoders. The encoder is also able to recover U_1 from \mathcal{C}_{s_1} . Next, the encoder applies lossy source coding to the sources (X, U_1) with reconstruction U_2 and targets distortions $d(X, U_2) = D_2$, $d(U_1, U_2) = \eta$. We then construct the code $\mathcal{C}_{s_2} = \mathcal{C}_N(F_{s_2}, \bar{0})$. Finally, the encoder applies channel coding to the symmetric test channel T_c with input U_2 and output (Y, U_1) . We derive code $\mathcal{C}_{c_2} = \mathcal{C}_N(F_{c_2}, u_{F_{c_2}}(\bar{v}))$, where $u_{F_{c_2}}(\bar{v})$ is defined by $u_{F_{s_2}} = \bar{0}$ and $u_{F_{c_2} \setminus F_{s_2}} = \bar{v}$ for $\bar{v} \in \{0, 1\}^{|F_{c_2} \setminus F_{s_2}|}$. The encoder sends the sub-vector $u_{F_{c_2} \setminus F_{s_2}}$ to the decoders.

Decoding: Decoder 1 receives $w_{F_{s_1}^c}$ and outputs the reconstruction sequence $U_1 = w_1^N G_N$. Decoder 2 receives $u_{F_{c_2} \setminus F_{s_2}}$, and hence, it can derive $u_{F_{c_2}}$. Moreover, Decoder 2 can also recover U_1 from $w_{F_{s_1}^c}$. Decoder 2 applies the SC decoding algorithm to obtain the codeword U_2 from the realizations of (Y, U_1) .

Next we present the rates that can be achieved in the proposed scheme. From the polarization theorem for lossy source coding in [10], we know that the required rate for Decoder 1 is $\frac{|F_{s_1}^c|}{N} \xrightarrow{N \rightarrow \infty} I(U_1; X) = 1 - h(D_1)$.

From the polarization theorems for source and channel coding [6], we obtain the rate for Decoder 2 as follows:

$$\begin{aligned} &\frac{|F_{c_2}| - |F_{s_2}|}{N} \xrightarrow{N \rightarrow \infty} I(U_2; X, U_1) - I(U_2; Y, U_1) \\ &= I(U_2; X) + I(U_1; X, U_2) - I(U_1; X) \\ &\quad - (I(U_2; Y) + I(U_1; Y, U_2) - I(U_1; Y)) \\ &= G(D_2) - G(D_1). \end{aligned}$$

Therefore, the total rate will be asymptotically given by

$$\frac{|F_{s_1}^c| + |F_{c_2}| - |F_{s_2}|}{N} \xrightarrow{N \rightarrow \infty} 1 - h(D_1 * p) + G(D_2),$$

for the region defined in Corollary 4.

Furthermore, the expected distortions are asymptotically approaching D_1 and D_2 at Decoders 1 and 2, respectively, when N is sufficiently large, according to [6], [10]. The encoding and decoding complexity of this scheme is $O(N \log N)$.

In our scheme, the performance of Decoder 2 is more challenging than that of Decoder 1. Hence, the simulation is conducted by fixing $D_1 = 0.35$, $p = 0.4$, and varying $D_2 \in (0, \min(d_c, D_1))$. These settings satisfy the requirements for the region in Corollary 4. The performance curves are shown in Fig. 3 for $n = 10, 12, 14, 16, 18$. It shows that the performance achieved by polar codes approaches the HBRD function as n increases.

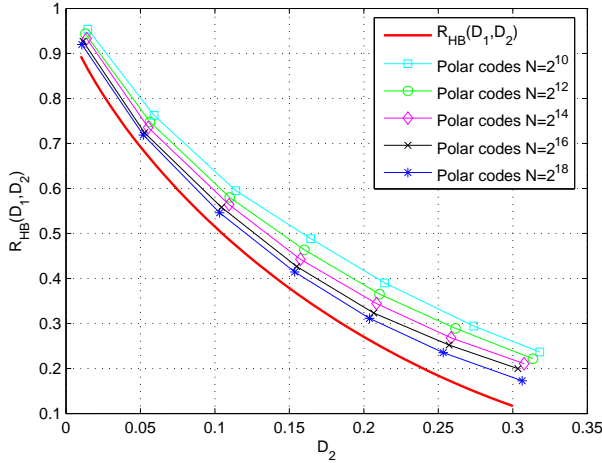


Fig. 3. Simulation performance of $R_{HB}(D_1, D_2)$ corresponding to D_2 .

Next, we present a coding scheme using polar codes for the rest of Region I. The explicit calculation of HBRD function for DSBS in Region I is given in [4] as follows:

Define the function

$$S_{D_1}(\alpha, \beta, \theta, \theta_1) \triangleq 1 - h(D_1 * p) + (\theta - \theta_1)G(\alpha) + \theta_1 G(\beta) + (1 - \theta)G(\gamma),$$

$$\text{where } \gamma = \begin{cases} \frac{D_1 - (\theta - \theta_1)(1 - \alpha) - \theta_1 \beta}{1 - \theta} & \theta \neq 1 \\ 0.5 & \theta = 1 \end{cases}, \text{ on the domain } 0 \leq \theta_1 \leq \theta \leq 1, 0 \leq \alpha, \beta \leq p, p \leq \gamma \leq 1 - p.$$

The following theorem characterizes the HBRD function in Region I.

Theorem 7. [4, Theorem 2] For $0 \leq D_1 < 0.5$ and $0 \leq D_2 < \min(D_1, p)$, $R_{HB}(D_1, D_2) = \min S_{D_1}(\alpha, \beta, \theta, \theta_1)$, where the minimization is over the domain of $S_{D_1}(\alpha, \beta, \theta, \theta_1)$, subject to the constraint $(\theta - \theta_1)\alpha + \theta_1\beta + (1 - \theta)p = D_2$.

The forward test channel structure is also given in [4] as reproduced in Table I. It constructs random variables with joint distribution $P_{X, U_1, U_2}(x, u_1, u_2)$, and the rate $I(X; U_1) +$

$I(X; U_2|U_1, Y) = S_{D_1}(\alpha, \beta, \theta, \theta_1)$. In this test channel structure, U_2 is a ternary random variable, i.e., $U_2 = \{0, 1, 2\}$.

We express U_2 as two binary random variables U_{21} and U_{22} , where $U_2 = 2U_{22} + U_{21}$, i.e., $U_2 = (0, 1, 2) \mapsto (U_{21}, U_{22}) = (00, 10, 01)$. For the coding scheme, we apply the same scheme specified previously for Decoder 1 to achieve D_1 . Again, U_1 and Y can be considered as side information for Decoder 2. The rate is evaluated by $I(X; U_2|U_1, Y) = I(X; U_{21}, U_{22}|U_1, Y) = I(X; U_{21}|U_1, Y) + I(X; U_{22}|U_1, U_{21}, Y)$. This means that we can design two coding schemes to achieve the rates $I(X; U_{21}|U_1, Y)$ and $I(X; U_{22}|U_1, U_{21}, Y)$, accordingly.

Moreover, we have $I(X; U_{21}|U_1, Y) = I(U_{21}; X, U_1, Y) - I(U_{21}; U_1, Y) = I(U_{21}; X, U_1) - I(U_{21}; U_1, Y)$, based on the Markov chain $Y \leftrightarrow X \leftrightarrow (U_1, U_{21}, U_{22})$. According to this Markov chain, we know that the test channel $T_{C_{21}}: U_{21} \rightarrow (Y, U_1)$ is degraded with respect to $T_{S_{21}}: U_{21} \rightarrow (X, U_1)$. Similarly to the coding scheme for the region of Corollary 4, we are able to construct a nested polar code structure to achieve $I(X; U_{21}|U_1, Y)$. Likewise, we can also achieve the rate $I(X; U_{22}|U_1, U_{21}, Y)$ following the same arguments.

Note that we may need to use polar codes developed for asymmetric models [7], since U_{21} and U_{22} may not be uniform. It is shown in [7] that polar codes are optimal for both source and channel coding in asymmetric models. Hence, this coding scheme can achieve the optimal HBRD function, as long as the optimal parameters α, β, θ , and θ_1 that achieve the minimum value of the function $S_{D_1}(\alpha, \beta, \theta, \theta_1)$ can be specified. The detailed coding scheme is skipped here due to the space constraint.

IV. POLAR LATTICES FOR GAUSSIAN SOURCES

It is shown in [8] that polar lattices can achieve the optimal rate-distortion performance for both the classical and the Wyner-Ziv compression for Gaussian sources under squared-error distortion. The Wyner-Ziv problem for the Gaussian case can be solved by a nested code structure that combines the AWGN capacity achieving polar lattices [11] and the rate-distortion optimal ones [8]. Here we show that the HBRD function for Gaussian sources can also be achieved by a similar nested code structure. In this section, we only present the basic design principle for the non-degenerate first region specified in (2) due to the page limitation.

We start with a basic introduction to polar lattices. An n -dimensional lattice is a discrete subgroup of \mathbb{R}^n which can be described by $\Lambda = \{\lambda = \mathbf{B}z : z \in \mathbb{Z}^n\}$, where \mathbf{B} is the full rank generator matrix. For $\sigma > 0$ and $c \in \mathbb{R}^n$, the Gaussian distribution of variance σ^2 centered at c is defined as $f_{\sigma, c}(x) = \frac{1}{(\sqrt{2\pi}\sigma)^n} e^{-\frac{\|x - c\|^2}{2\sigma^2}}$, $x \in \mathbb{R}^n$. Let $f_{\sigma, 0}(x) = f_{\sigma}(x)$ for short. The Λ -periodic function is defined as $f_{\sigma, \Lambda}(x) = \sum_{\lambda \in \Lambda} f_{\sigma, \lambda}(x) = \frac{1}{(\sqrt{2\pi}\sigma)^n} \sum_{\lambda \in \Lambda} e^{-\frac{\|x - \lambda\|^2}{2\sigma^2}}$. We note that $f_{\sigma, \Lambda}(x)$ is a probability density function (PDF) if x is restricted to the fundamental region $\mathcal{R}(\Lambda)$. It is actually the PDF of the Λ -aliased Gaussian noise [12].

	$(u_1, x) = (0, 0)$	$(u_1, x) = (0, 1)$	$(u_1, x) = (1, 0)$	$(u_1, x) = (1, 1)$
$u_2 = 0$	$0.5\theta_1(1 - \beta)$	$0.5\theta_1\beta$	$0.5(\theta - \theta_1)(1 - \alpha)$	$0.5(\theta - \theta_1)\alpha$
$u_2 = 1$	$0.5(\theta - \theta_1)\alpha$	$0.5(\theta - \theta_1)(1 - \alpha)$	$0.5\theta_1\beta$	$0.5\theta_1(1 - \beta)$
$u_2 = 2$	$0.5(1 - \theta)(1 - \gamma)$	$0.5(1 - \theta)\gamma$	$0.5(1 - \theta)\gamma$	$0.5(1 - \theta)(1 - \gamma)$

TABLE I
JOINT DISTRIBUTION $P_{X,U_1,U_2}(x, u_1, u_2)$ [4].

The flatness factor of a lattice Λ is defined as $\epsilon_\Lambda(\sigma) \triangleq \max_{x \in \mathcal{R}(\Lambda)} |V(\Lambda)f_{\sigma,\Lambda}(x) - 1|$, where $V(\Lambda) = |\det(\mathbf{B})|$ denotes the volume of a fundamental region of Λ [12]. It can be interpreted as the maximum variation of $f_{\sigma,\Lambda}(x)$ with respect to the uniform distribution over a fundamental region of Λ .

We define the discrete Gaussian distribution over Λ centered at c as the discrete distribution taking values in $\lambda \in \Lambda$ as $D_{\Lambda,\sigma,c}(\lambda) = \frac{f_{\sigma,c}(\lambda)}{f_{\sigma,c}(\Lambda)}$, $\forall \lambda \in \Lambda$, where $f_{\sigma,c}(\Lambda) = \sum_{\lambda \in \Lambda} f_{\sigma,c}(\lambda)$. For convenience, we write $D_{\Lambda,\sigma} = D_{\Lambda,\sigma,0}$. It has been shown in [13] that lattice Gaussian distribution preserves many properties of the continuous Gaussian distribution when the flatness factor is negligible. To keep the notations simple, we always set $c = 0$ and $n = 1$ (one-dimensional lattice Λ) in this work.

For the Gaussian Heegard-Berger problem, let $(X, Y, Z, Z_1, Z_2, U_1, U_2)$ be chosen as specified in Section II-C. Given Y as the side information for Decoder 2, the rate-distortion bound for the first region is given in the first line of (2).

To achieve the rate-distortion bound for Decoder 1, we can design a polar lattice for quantization for source X with variance σ_X^2 and distortion D_1 according to [8]. Further, both Decoder 1 and 2 can reconstruct U_1 with an average distortion asymptotically close to D_1 for any rate $R_1 > \frac{1}{2} \log(\sigma_X^2/D_1)$ [8]. Therefore, (U_1, Y) can be regarded the side information of Decoder 2.

To achieve the bound for Decoder 2, we firstly design a quantization polar lattice L_1 for source $X - U_1$ with Gaussian reconstruction alphabet U'_2 . In fact, $X - U_1 = Z_1 \sim \mathcal{N}(0, D_1)$ is Gaussian and independent of U_1 . Letting $\gamma = \frac{D_1\sigma_Z^2}{D_1\sigma_Z^2 - D_2(D_1 + \sigma_Z^2)}$, the distortion between $X - U_1$ and U'_2 is targeted to be $D'_2 = \gamma D_2$, i.e., $U'_2 = X - U_1 + N'_1$, where $N'_1 \sim \mathcal{N}(0, \gamma D_2)$. Moreover, we know that $Y - U_1 = X - U_1 + Z$. Then we can apply the minimum mean square error (MMSE) rescaling parameter $\alpha = \frac{D_1}{D_1 + \sigma_Z^2}$ to $Y - U_1$. As a result, we obtain $X - U_1 = \alpha(Y - U_1) + Z_3$, where $Z_3 \sim \mathcal{N}(0, \alpha\sigma_Z^2)$. We can also write $U'_2 = \alpha(Y - U_1) + N'_2$, where $N'_2 \sim \mathcal{N}(0, \gamma D_2 + \alpha\sigma_Z^2)$, which requires an AWGN capacity achieving polar lattice L_2 from $\alpha(Y - U_1)$ to U'_2 . The final reconstruction of Decoder 2 is given by $\hat{X}_2 = U_1 + \alpha(Y - U_1) + \frac{1}{\gamma}(U'_2 - \alpha(Y - U_1))$. We can check that \hat{X}_2 achieves distortion D_2 . The rate for Decoder 2 is then given by $R_2 > I(U'_2; X - U_1) - I(U'_2; \alpha(Y - U_1)) = \frac{1}{2} \log\left(\frac{D_1\sigma_Z^2}{D_2(D_1 + \sigma_Z^2)}\right)$, taking advantage of the fact that the polar lattices L_1 and L_2 can be shaped simultaneously.

Note that, although U'_2 is a continuous Gaussian random variable, we can use the discrete Gaussian distribution $D_{\Lambda,\sigma_{U'_2}^2}$

to replace it. Before that, we need to derive the MMSE rescaled version of the test channel mentioned in the previous paragraph, according to [8]. When the flatness factor is negligible, the total rate $R_1 + R_2$ can be made arbitrarily close to $\frac{1}{2} \log\left(\frac{\sigma_X^2\sigma_Z^2}{D_2(D_1 + \sigma_Z^2)}\right)$, which is the same as the first bound given in (2). From the test channels for L_1 and L_2 , it is not difficult to observe that L_2 is nested within L_1 .

V. CONCLUSIONS

We have presented rate-distortion optimal polar codes for the HBRD problem. In particular, for DSBS, we provided an explicit nested construction of polar codes that can achieve the optimal HBRD function for non-degenerate subset of distortion pairs. For the Gaussian setting, we have proposed a nested construction of polar lattices that combines quantization and AWGN capacity-achieving polar lattices, and achieves the optimal HBRD function.

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